As a simple example, let L be a complex symmetric operator and let

$$h = h_1 + \alpha h_2, \quad (1)$$

where h_1 and h_2 are any two trial functions and α is a complex parameter. Then

$$\frac{(\bar{h}, Lh)}{(\bar{h}, h)} = \frac{(\bar{h}_1, Lh_1) + 2\alpha(\bar{h}_1, Lh_2) + \alpha^2(\bar{h}_2, Lh_2)}{(\bar{h}_1, h_1) + 2\alpha(\bar{h}_1, h_2) + \alpha^2(\bar{h}_2, h_2)}. \quad (2)$$

But the right-hand side of (2) is the quotient of two quadratic polynomials in α, and in general it will attain any preassigned value exactly twice as α ranges over the whole complex plane.[8] For example, it can be made equal to any eigenvalue of L, even though h_1 and h_2 are entirely unrelated to the corresponding eigenfunction. On the other hand, applying the Rayleigh-Ritz procedure to h_1 and h_2 yields two values of α for which the corresponding values of $(\bar{h}, Lh)/(\bar{h}, h)$ need not and generally do not coincide with any eigenvalue of L.

A numerical example shows the sort of thing that can happen. Let

$$Lv \equiv \int_{-1}^{1} [1 + i\tfrac{1}{2}(x + y)]v(y)dy, \quad (3)$$

and choose the trial functions

$$h_1(x) = 1 - sx^2 + isx^3,$$
$$h_2(x) = 1 + i2x, \quad (4)$$

where s is a real constant to be specified presently. Straightforward calculation gives

$$(\bar{h}_1, Lh_1) = 4 - \frac{52s}{15} + \frac{32s^2}{45},$$
$$(\bar{h}_1, Lh_2) = \frac{8}{3} - \frac{58s}{45},$$
$$(\bar{h}_2, Lh_2) = \frac{4}{3},$$
$$(\bar{h}_1, h_1) = 2 - \frac{4s}{3} + \frac{4s^2}{35},$$
$$(\bar{h}_1, h_2) = 2 - \frac{22s}{15},$$
$$(\bar{h}_2, h_2) = -\frac{2}{3}. \quad (5)$$

We now choose s to be the larger of the two roots of the equation

$$\frac{(\bar{h}_1, Lh_1)}{(\bar{h}_1, h_1)} = 1 + \tfrac{1}{3}\sqrt{6}. \quad (6)$$

To five decimals,

$$s = 1.62669. \quad (7)$$

The determinantal equation resulting from the Rayleigh-Ritz process [Kaplan's (10)] becomes, numerically,

$$\begin{vmatrix} 0.24250-0.13350\lambda & 0.57005+0.38581\lambda \\ 0.57005+0.38581\lambda & 1.33333+0.66667\lambda \end{vmatrix} = 0. \quad (8)$$

[8]Notice that the variational quotient associated with a *Hermitian* operator, as given by (4) of Morgan,[1] cannot generally be made to assume arbitrary values by proper choice of α.

The roots of this equation and the corresponding "eigenfunctions" are

$$\tilde{\lambda}_1 = -0.00357, \quad \tilde{v}_1 = h_1 - 0.42727h_2,$$
$$\tilde{\lambda}_2 = -1.91444, \quad \tilde{v}_2 = h_1 + 2.95501h_2. \quad (9)$$

But since the kernel of the operator L is of finite rank, it is easy to calculate the eigenvalues and eigenfunctions exactly. They are[9]

$$\lambda_1 = 1 + \tfrac{1}{3}\sqrt{6} \approx 1.81650,$$
$$v_1 = 1 + i(3 - \sqrt{6})x \approx 1 + i0.55051x;$$
$$\lambda_2 = 1 - \tfrac{1}{3}\sqrt{6} \approx 0.18350,$$
$$v_2 = 1 + i(3 + \sqrt{6})x \approx 1 + i5.44949x. \quad (10)$$

Comparing (6) and (10), we see that the eigenvalue λ_1 coincides exactly with the Rayleigh quotient $(\bar{h}_1, Lh_1)/(\bar{h}_1, h_1)$, even though h_1 is not a multiple of v_1. However the Rayleigh-Ritz procedure gives no indication of this, and when applied to the trial functions h_1 and h_2 it yields "eigenvalues" $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ which are much worse than we would have obtained from h_1 alone. Hence the procedure clearly does not give the best approximation to λ_1 which can be had from a linear combination of h_1 and h_2.

This innocent-looking example shows that the use of the Rayleigh-Ritz procedure to refine an approximate eigenvalue of a complex symmetric operator can actually lead to a worse approximation than the initial one. One might conjecture, of course, that even though the variational method does not approximate eigenvalues very well, it does give an optimal approximation for the eigenfunctions. Unfortunately this is not true either. If one defines the distance between two complex-valued functions in terms of a quadratic metric, then the best approximation to v_1 which can be obtained using a linear combination of h_1 and h_2 is found by minimizing

$$I(\rho_1, \rho_2) = (v_1 - \rho_1 h_1 - \rho_2 h_2, v_1 - \rho_1 h_1 - \rho_2 h_2) = \int_a^b |v_1 - \rho_1 h_1 - \rho_2 h_2|^2 dx \quad (11)$$

with respect to the complex coefficients ρ_1 and ρ_2. The minimization is easily carried out in the present example where v_1 is known, but it does not lead to either of the Rayleigh-Ritz "eigenfunctions."

Dr. Kaplan suggests that the key to the successful use of variational procedures in non-self-adjoint problems is the selection of appropriate trial functions, in the light of experience and knowledge of the physical process. One can hardly quarrel with this objective, although it may be easier to carry out when the unknown functions are real than when they are complex. Certainly no law prohibits anyone from formally applying the Rayleigh-Ritz procedure to a selected set of functions and thinking that he has obtained a better approximation by so doing. Undoubtedly in some cases he actually will get a better approximation than he started with; but if he does not know the exact eigenvalue in advance, he can never be quite sure whether the Rayleigh-Ritz procedure has served him well or ill.

[9] Zero is also an eigenvalue of L of infinite multiplicity, but it plays no role in the present argument.

To the best of my knowledge, no theorems have ever been proved which specify the conditions, if any, under which the Rayleigh-Ritz procedure will yield an improved approximation to an eigenvalue of a non-self-adjoint (for example, complex symmetric) operator. There *are* such theorems for self-adjoint or Hermitian operators. Whether anything similar can be obtained for nonself-adjoint operators is an interesting open question.

I should like to acknowledge a stimulating exchange of correspondence on this subject with Professor R. F. Harrington of Syracuse University.

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Comment on "Broadband Microwave Discriminator"

The frequency discriminator described by R. J. Mohr[1] is capable of a simple extension[2] to give a device with important advantages and many practical applications.

Mohr's circuit (Fig. 1) measures frequency in the form

$$\frac{|E_2|^2}{|E_1|^2} = \tan^2 \pi l \frac{f}{c}$$

where l is the length of the phase delay line, f is the frequency, c the phase velocity and

$$\frac{2\pi l f}{c} = \phi.$$

An alternative method of processing the detected signals is to take the difference

$$|E_1|^2 - |E_2|^2 = E^2 \cos 2\pi l \frac{f}{c},$$

and by duplicating the circuit so that $\phi' = \phi - (\pi/2)$, we have

$$|E_3|^2 - |E_4|^2 = E^2 \sin 2\pi l \frac{f}{c}$$

and hence

$$\frac{|E_3|^2 - |E_4|^2}{|E_1|^2 - |E_2|^2} = \tan 2\pi l \frac{f}{c} = \tan \phi.$$

Simply, a $\lambda/4$ length of line may be used to subtract $\pi/2$ from the phase delay ϕ and Fig. 2 shows a typical circuit. Some nonlinearity in the frequency characteristic will result from the fact that the phase change of $\pi/2$ will vary with frequency However, this is small and circuits operating over frequencey ranges up to 6:1 in the band 0.15 Gc to 11.5 Gc have been successfully used. An absolute measuring accuracy of $\pm 5°$ in ϕ is typical and may be improved by calibrating individual circuits.

Manuscript received December 9, 1963.
[1] R. J. Mohr, "Broadband microwave discriminator," IEEE Trans. on Microwave Theory and Techniques, vol. 11, p. 263; July, 1963.
[2] S. J. Robinson, "Microwave Frequency Measuring Device," British Patent Application, No. 22471/58; July, 1958.

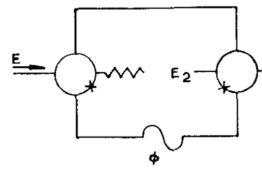

Fig. 1—Simple frequency discriminator.

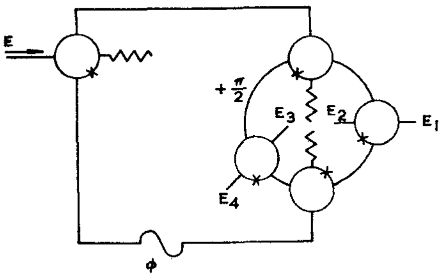

Fig. 2—Quadrature frequency discriminator.

Several advantages of the arrangement may be quoted:

1) An (r, θ) display can be used so that $\theta \propto \phi \propto f$.
2) Using an (r, θ) display, θ continuously and linearly increases with frequency over any range, although, of course, ambiguities occur in a range of ϕ exceeding 2π. "Clock" systems to give increased accuracy without ambiguity are possible. In this context, frequency ranges representing $2n\pi < \phi < 2(n+1)\pi$ of 10 Mc and 10,000 Mc are equally practicable.
3) $|E_1|^2 - |E_2|^2 \simeq |E_1| - |E_2|$ when $E_1 = \cos \phi/2$ and $E_2 = \sin \phi/2$ so that errors due to departure from square law in the detector characteristics are very small.
4) The subtraction $|E_1|^2 - |E_2|^2$ may be written

$$(E'^2 + E''^2 + 2E'E'' \cos \phi) - (E'^2 + E''^2 - 2E'E'' \cos \phi) = 4E'E'' \cos \phi$$

where E' and E'' are the input voltages to a phase-measuring hybrid junction, and $E' \neq E''$. It may be shown that the product $(E'E'')$, for both the phase measuring junctions, shown in Fig. 2, is not dependent upon equality of power split in the power dividing junctions and that therefore this equality is not necessary for good frequency measuring performance.

It should be noted that the four-junction circuit, giving the sine and cosine terms, is the same as that for a single-sideband modulator and is one of a large family of multiport networks that might be used in phase comparison applications. For example, an eight detector device giving $\cos \phi$, $\cos(\phi + \pi/4)$, $\cos(\phi + \pi/2)$ and $\cos(\phi + 3\pi/4)$ outputs can easily be realized. Such an arrangement shows improved measuring accuracy by removing quadrantal error terms.

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Harmonic Generation by an Array

In the millimeter region, harmonic generators have long served as convenient signal sources. However, the power handling capacity of the diode elements is very limited. At the short-wave end of the millimeter spectrum, the dominant mode waveguide terminal of a harmonic generator may also be less desirable than a quasi-optical or beam output. These factors have led to the investigation of a diode array as a millimeter wave source. The results obtained show that such an array is feasible but uneconomic with presently available diodes.

A schematic diagram of a harmonic array is shown in Fig. 1. The fundamental power illuminates an array of receiving apertures from a feed horn which may be extended (as shown by the broken lines) to shield the entire input region, if desired. Depending on the spacing between feed and array, it may be necessary to introduce phase correction by means of a lens, or by changing the lengths of input waveguide in each multiplier unit. These units consist of a receiving or input horn coupled to an inline harmonic generator. The output guide is proportioned to pass only the desired harmonic and higher terms which are neglected. Each output horn occupies the same cross section as the corresponding input aperture. This provides grating lobe suppression, since a narrower element pattern compensates for the wider spacing at the output frequency. The output is in the form of a beam, which can be brought to a focus by choosing the proper phase correction on either the input or output side of the array.

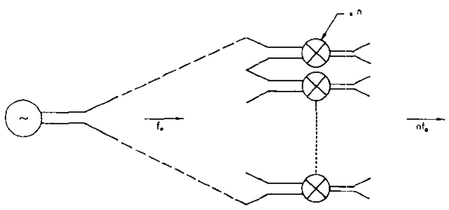

Fig. 1—Harmonic array.

To investigate the properties of a harmonic array, a 2×5 element array was constructed for $f_0 = 25$ Gc, $n = 2$. A very simple straight-through element, as shown in Fig. 2, was used. The individual crystals produced an input VSWR between 4:1 and 6:1 in this mount. Measurements on individual elements showed that the standard deviations in insertion phase shift and conversion loss were less than 25° and 1.7 db respectively for the 27 individual 1N26 crystals tested. The harmonic output of the array was within 1 db of the output calculated for the sum of the individual elements, each with the measured VSWR, and illumination corrections applied. It was therefore concluded that a harmonic array functions as a true additive structure. The array is shown in Fig. 3.

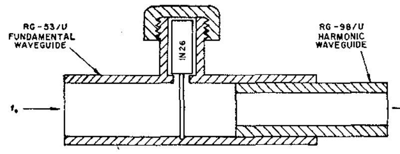

Fig. 2—Harmonic array element.

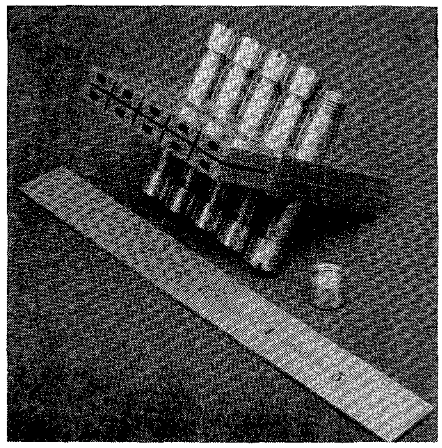
Fig. 3—Ten element harmonic array.

The performance of the experimental array shows conversion and coupling losses of about 20 db and 6 db respectively. With provisions for impedance matching in the elements, and with more efficient harmonic generators, a useful array source for millimeter and possibly submillimeter power might be constructed.

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Analogous Propagation Modes in Inhomogeneous Plasma and Tapered Waveguide

An interesting analogy exists between the propagation of transverse electromagnetic (TE) waves in a plasma (with no magnetic field) and in conventional waveguide.[1,2] This analogy reflects the similar roles played by the volume conduction current in the plasma and the wall conduction current in the waveguide and is of interest in that it provides insight into plasma propagation and suggests the possibility of simulating

Manuscript received July 19, 1963; revised December 16, 1963. This work was performed under the auspices of the U. S. Atomic Energy Commission.

[1] V. L. Ginzburg, "Propagation of Electromagnetic Waves in Plasma," Gordon and Breach Publishers, Inc., New York, N. Y.; 1962.

[2] W. Rotman, "Plasma simulation by artificial dielectrics and parallel-plate media," IRE Trans. on Antennas and Propagation, vol. AP-10, pp. 82–95; January, 1962.